Stop Wasting My Gradient: A Practical SVRG

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Minimizing Finite Sum

- We want to minimize the sum of a finite set of smooth functions

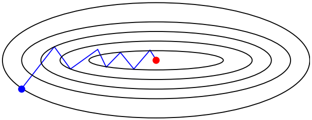
$$\min_{x \in \mathbb{R}^d} f(x) := \frac{1}{n} \sum_{i=1}^{n} f_i(x)$$

- We are interested in cases where n is very large.
- We will focus on strongly-convex functions
- Simplest example is l2-regularized least-squares
  $f_i(x) = (a_i^T x - b_i)^2 + \frac{\lambda}{2} \|x\|^2$
- Common framework in data fitting problem
    - logistic regression, Huber regression, smooth SVMs, CRFs, etc.

Stochastic vs. Deterministic Gradient Methods

- Deterministic gradient method [Cauchy, 1847]:

- $X_{t+1} = X_t - \alpha_t f'(X_t) = X_t - \frac{\alpha_t}{n} \sum_{i=1}^{n} f_i'(X_t)$
- Linear convergence rate
- Iteration cost is linear in n



Stochastic vs. Deterministic Gradient Methods

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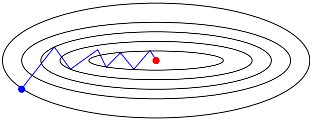
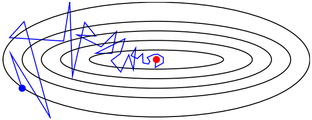


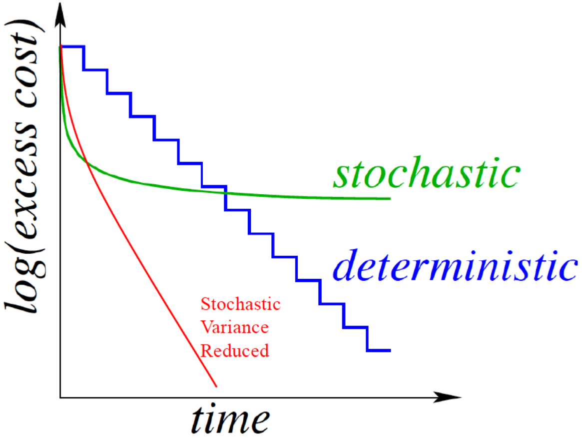
- Stochastic gradient method [Robins and Monro, 1951]:

- Randomly pick i_t in iteration t from $\{1, \ldots, n\}$
  $X_{t+1} = X_t - \alpha_t f_{i_t}'(X_t)$
- Iteration cost is independent of n
- Sub-linear convergence rate



Motivations for New Methods

- Stochastic Variance Reduced Methods: Linear convergence rate + O(1) iteration cost



Motivations for New Methods

- SAG [Le Roux et.al 2012]
- SDCA [Shalev-Shwartz and Zhang, 2013]
- MISO [Marial, 2013]
- SAGA [Defazio, et al.,2014]
- These methods all need memory to store gradient of f_i's or dual variable
    - $O(nd)$ space for general objective function.

Stochastic Variance-Reduced Gradient (SVRG)

- Recent methods with similar rates that avoid memory:
  - Mixed Gradient [Mahdavi & Jin, 2013, Zhang et al., 2013]
  - Stochastic variance-reduced gradient (SVRG) [Johnson & Zhang, 2013]
  - Semi-stochastic gradient [Konecny & Richtarik, 2013]
- Memory is only O(d), but they require extra gradient calculations:
  - Two gradients on each iteration.
  - Occasional calculation of all n gradients.
- Extra calculations make them slower than SAG and friends.

Outline

1. Deterministic, stochastic, and finite-sum methods
2. Wasting fewer gradients in SVRG
3. Some Heuristic For SVM
4. Conclusion

Stochastic Variance-Reduced Gradient

SVRG Algorithm(m, α,x_0)

- start with x_0
  - for $t = 0, 1, \ldots, m$
    - randomly pick i_t
      $x^{t+1} = x^t - \alpha(f'_{i_t}(x^t))$

Stochastic Variance-Reduced Gradient

SVRG Algorithm(m, α,x_0)

- start with x_0
  - for $t = 0, 1, \ldots, m$
    - randomly pick i_t
      $x^{t+1} = x^t - \alpha(f'_{i_t}(x^t) - f'_{i_t}(x_s) + d_s)$ (two gradients per iteration)

Stochastic Variance-Reduced Gradient

SVRG Algorithm(m, α,x_0)

- start with x_0
- for $s = 0, 1, 2, \ldots$ (outer loop)
  $d_s = \frac{1}{n}\sum_{i=1}^{n} f_i'(x_s)$ (full gradient evaluation)
  - $x^0 = x_s$
  - for $t = 0, 1, \ldots, m$ (inner loop)
    - randomly pick i_t
      $x^{t+1} = x^t - \alpha(f_{i_t}'(x^t) - f_{i_t}'(x_s) + d_s)$ (two gradients per iteration)
  - $x_{s+1} = x^t$ for a random $t \in \{1, \ldots, m\}$

Convergence Analysis of SVRG

- Assumptions:
  - Each f_i is convex.
  - Each ∇f_i is L-Lipschitz continuous.
  - f is μ-strongly convex.
- Johnson & Zhang [2013] show that outer loop satisfies
  $\mathbb{E}[f(x_{s+1}) - f(x^*)] \leq \rho[f(x_s) - f(x^*)], \;\; \rho = \frac{1}{1-2\alpha L}(2L\alpha + \frac{1}{m\mu\alpha})$
- SVRG rate is very fast for appropriate step size α and inner-loop size m.
- In practice: $m = n, \;\; \alpha = 1/L, \;\; x_{s+1} = x^m$

Convergence Analysis of SVRG with Error

- Assume:
  - We approximate full gradient by $d^s = f'(x_s) + e^s$
  - $\|x^t - x^*\| \leq Z$ for some Z
- Then SVRG with error satisfies

$$E[f(x_{s+1}) - f(x^*)] \leq \rho[f(x_s) - f(x^*)] + \frac{\alpha\mathbb{E}\left[\|e^s\|^2\right] + Z\mathbb{E}\left[\|e^s\|\right]}{1 - 2\alpha L}$$

- Implications
  - faster rate when far from solution.
  - Same convergence rate if $\max\{\mathbb{E}\left[\|e^s\|\right], \mathbb{E}\left[\|e^s\|^2\right]\} = O(\tilde{\rho}^s)$ for $\tilde{\rho}^s \leq \rho$

Reducing Gradient Evaluations with Batching

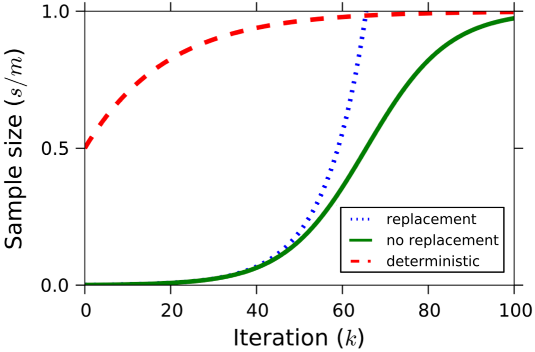
- SVRG requires $2m + n$ gradients for each m iterations.
- We can reduce the n by using a mini-batch $\mathcal{B}^s$ of training examples

$$d^s = \frac{1}{|\mathcal{B}^s|} \sum_{i \in \mathcal{B}^s} f_i'(x_s)$$

- Special case of SVRG with error, batch size controls error.

$$|\mathcal{B}^s| \geq \frac{nS^2}{S^2 + n\gamma\tilde{\rho}^{2s}}$$

[Aravkin et al, 2012]



Batching SVRG Algorithm

Algorithm 1 Batching SVRG

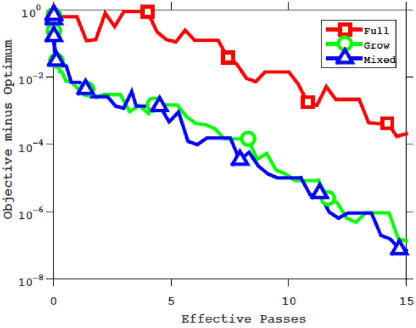
Input: initial vector x^0, update frequency m, learning rate α.
for $s = 0, 1, 2, \ldots$ **do**
  $\mathcal{B}^s$=$|\mathcal{B}^s|$ elements sampled without replacement from $\{1, 2, \ldots, n\}$.
  $d^s = \frac{1}{|\mathcal{B}^s|} \sum_{i \in \mathcal{B}^s} f_i^{'}(x^s)$
  x^0=x_s
  for $t = 1, 2, \ldots, m$ **do**
    Randomly pick $i_t \in 1, \ldots, n$
    $x^{t+1} = x^t - \alpha(f'_{i_t}(x^t) - f'_{i_t}(x_s) + d^s)$
  end for
  option I: set $x_{s+1} = x^m$
  option II: set $x_{s+1} = x^t$ for a random $t \in \{1, \ldots, m\}$
end for

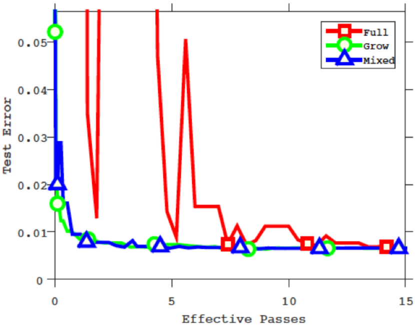
- Growing-batch reduces n in the $2m + n$ cost of SVRG.
- But does not improve the 2
- Mixing SGD with SVRG

Numerical Experiments with Batching

- Training/testing loss for ‡∈-regularized logistic on spam filtering data.

$$\underset{x \in \mathbb{R}^d}{\arg\min} \frac{\lambda}{2}\|x\|^2 + \frac{1}{n}\sum_{i=1}^{n} \log(1 + exp(-b_i a_i' x))$$





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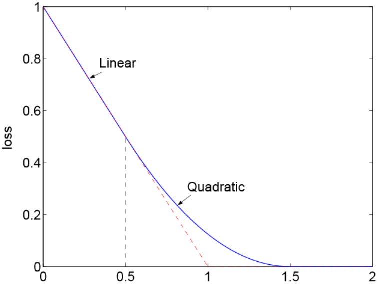
Identifying Support Vectors

- Mixed strategy improves error when far from solution.
- For certain objectives, can improve close to solution.
- Consider Huberized hinge loss problem [Rosset & Zhu, 2006]

$$\min_{x \in \mathbb{R}^d} \frac{1}{n} \sum_{i=1}^{n} f(b_i a_i' x),$$

$$f(\tau) = \begin{cases} 0 & \text{if} \tau > 1 + \epsilon \\ 1 - \tau & \text{if} \tau < 1 - \epsilon \\ \frac{(1+\epsilon-\tau)^2}{4\epsilon} & \text{if} |1 - \tau| \leq \epsilon \end{cases}$$



- The solution is sparse in the f_i' (has support vectors).

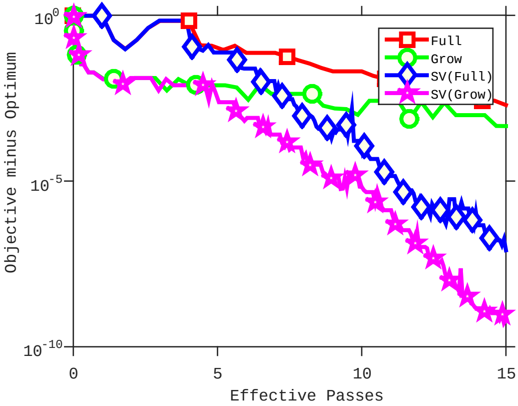
Using Support Vectors

- Non-support examples do not contribute to solution
- We can skip gradient evaluations where we expected/know that $f_i' = 0$
- Approach 1: sound pruning
  - Maintain list of support vectors at x_s.
  - Do not evaluate $f_i(x_s)$ if it is not a support vector.
  - Can reduce number of gradients per iteration to 1.

Using Support Vectors

- Non-support examples do not contribute to solution
- We can skip gradient evaluations where we expected/know that $f_i' = 0$
- Approach 2: heuristic pruning
  - Keep track of the number of times we $f_i'(x_s) = 0$ or $f_i'(x^t) = 0$.
  - If it continues to be zero, skip its next 2 evaluations.
  - If it continues to be zero, skip its next 4 evaluations.
  - If it continues to be zero, skip its next 8 evaluations.
  - Can reduce number of gradients per iteration to 1 exponentially.

- $\mathcal{L}2$-regularized Huberized hinge on spam filtering data.



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Conclusion

- Stochastic methods for minimizing finite sum with linear convergence
- SVRG is the only method without a memory requirement
- Reducing gradient evaluation by inexact full gradient
- A heuristic SVM algorithm
- Other variants and analysis
  - Mixed Strategy
  - Proximal SVRG
  - SVRG with non-uniform sampling
  - Fixed-Random Mini-Batching Strategy
  - Generalization error
- Thank you!